ON THE DEEP-WATER STOKES WAVE FLOW

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Abstract. We prove a new result detailing the monotonicity of the horizontal velocity component of deep-water Stokes waves along streamlines.

1. Introduction

An aspect of the theory of water waves which has generated much interest in recent years is the description of the movement of individual fluid particles as a surface wave propagates. As far back as the 19^{th} century, Stokes [25] observed that for a wave propagating along an expanse of water there was an average forward mass drift in the direction of the wave, the so-called Stokes drift. His observation related to an averaged motion of the water mass, however, and provided no information regarding the behaviour of individual water particles. Nonlinearities play a vital role in Stokes arguments, and it is interesting to note that in the literature, both classical and modern, it is commonly conjectured that in shallow water the trajectories of fluid particles in irrotational flow take the form of closed ellipses, while in deep-water the paths are circles (see [1, 13, 21, 22, 24]). Such arguments rely on a double approximation of the governing equations— the fully nonlinear equations are linearised using the shallow water approximation, and the solution of the linearised equations is then in turn further approximated to obtain a closed path. Such reasoning seems unsatisfactory however, and it was recently proved, using exact methods on the linearised equations, that the conjecture of closed particle paths is in fact false for linear water waves propagating both over a flat bed [12, 18] and in water which is infinitely deep [5, 19]. Subsequent work, undertaken within the nonlinear framework of the full governing equations for Stokes waves travelling over a flat bed, further showed that there are no closed particle paths for nontrivial waves [4, 8], and this work was extended to the deep water case in [20]. These results depends on the irrotationality of the Stokes flow— for rotational flow Gerstner's wave, described by Gerstner in 1809 [17], provides a solution of the full equations where the particle describes closed paths. Interestingly, this explicit solution of the full governing equations was later independently re-discovered by Rankine, see [23]; a rigorous mathematical analysis of these solutions is presented in [2], see also [3]. A discussion and some results describing the effects of vorticity on the particle paths can be found in [15], see also [6, 7, 9–11, 14, 27].

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It is the aim of this note, which might be considered an addendum to [20], to provide further results on the velocity field of the particles for the deep water Stokes wave, namely, by furnishing an exact description of the horizontal component of the velocity of the particles along the streamlines. This is achieved by introducing a conformal hodograph transform which enables us to perform harmonic analysis in the new variables. This new result simplifies considerably the approach used recently in [20] to show that all particle trajectories in a deep-water Stokes flow describe non-closed paths.

2. The governing equations

A deep-water Stokes wave is a two-dimensional periodic wave with a symmetric profile that rises and falls exactly once per wavelength and which is acted on by gravity, travelling at constant speed $c > 0$ on the surface of irrotational water which is infinitely deep. We formulate the governing equations for the Stokes wave in the frame moving with speed $c > 0$, and so the flow in this frame is steady with stationary surface profile. Let $\overline{D_\eta} = \{(x, y) \in \mathbb{R}^2 : -\infty < y \leq \eta(x)\}$ represent the closure of the fluid domain. If the velocity field in the moving frame is given by $(u(x, y), v(x, y))$, the governing equations for the deep water Stokes wave are [20]:

$$\Delta\psi = 0 \quad \text{in} \quad -\infty < y < \eta(x), \tag{2.1}$$

$$|\nabla\psi|^2 + 2gy = E_0 \quad \text{on} \quad y = \eta(x), \tag{2.2}$$

$$\psi = 0 \quad \text{on} \quad y = \eta(x), \tag{2.3}$$

$$\nabla\psi \to (0, -c) \quad \text{as} \quad y \to -\infty \text{ uniformly for } x \in \mathbb{R}, \tag{2.4}$$

where $\eta \in C^3(\mathbb{R})$ and $\psi \in C^2(\overline{D_\eta})$ are periodic in the $x-$variable, with wavelength λ as the period, and η rises and falls exactly once per period with $\eta'(x) \neq 0$ except at the maximum or minimum. The stream function ψ is defined, up to a constant, by

$$\psi_y = u - c, \quad \psi_x = -v. \tag{2.5}$$

The mean horizontal velocity per wavelength λ, at any fixed depth below the wave trough level, is constant throughout D_η for the deep water Stokes wave (see [20]). A natural consequence of this is Stokes' definition of the wave speed as the mean velocity in the moving frame of reference for which the wave is stationary,

$$c = -\frac{1}{\lambda} \int_0^\lambda \psi_y(x, y_0)dx > 0, \tag{2.6}$$

where y_0 is any fixed depth below the wave trough level.

3. Some results for the velocity field

Without any loss of generality we restrict our attention to Stokes waves of period 2π with the crest at $(0, \eta(0))$ and the trough at $(\pi, \eta(\pi))$.

Lemma 3.1. *[20] The following strict inequalities hold*

$$\psi_x(x,y) < 0, \quad \frac{d}{dx}u(x,\eta(x)) < 0 \quad \text{for} \quad x \in (0,\pi), y \in (-\infty, \eta(x)]. \tag{3.1}$$

Lemma 3.2. *[20] The function $y \mapsto u(0,y)$ is strictly decreasing as we move down along the vertical half-line $[(0,\eta(0)),(0,-\infty)]$, whereas it is strictly increasing along $[(\pi,\eta(\pi)),(\pi,-\infty)]$ as we move downwards.*

Remark 3.3. Prior to this we have assumed that $u < c$ in D_η. As a result of Lemma 3.1 and Lemma 3.2 we can now state that $u < c$ in the closure $\overline{D_\eta}$ of D_η, except in the case of a Stokes wave of greatest height, in which case at the crest $(0,\eta(0))$ we have $u = c$ with $u < c$ at all other points of $\overline{D_\eta}$ cf. [26].

The velocity potential for the motion, $\phi(x,y)$, is defined as

$$\phi_x := u - c = \psi_y, \quad \phi_y := v = -\psi_x,$$
$$\phi = 0 \text{ on } y = \eta(x).$$

By means of the stream function ψ and the velocity potential ϕ we can perform the hodograph transformation

$$\begin{cases} q = -\phi(x,y), \\ p = -\psi(x,y). \end{cases} \tag{3.2}$$

The hodograph transformation is conformal and it transforms the free boundary problem into a nonlinear fixed boundary problem for the harmonic function

$$h(q,p) = y + d$$

in a fixed semi-infinite rectangular domain. Following the transformation we have

$$\begin{cases} \partial_q = h_p\partial_x + h_q\partial_y, \\ \partial_p = -h_q\partial_x + h_p\partial_y, \end{cases} \tag{3.3}$$

and

$$\begin{cases} \partial_x = (c-u)\partial_q + v\partial_p, \\ \partial_y = -v\partial_q + (c-u)\partial_p, \end{cases} \tag{3.4}$$

while

$$h_q = -\frac{v}{(c-u)^2+v^2} = -\frac{\partial x}{\partial p} = \frac{\partial y}{\partial q}, \quad h_p = \frac{c-u}{(c-u)^2+v^2} = \frac{\partial x}{\partial q} = \frac{\partial y}{\partial p}. \tag{3.5}$$

We now use the conformal hodograph transformation to prove the following result.

Theorem 3.4. *Along every streamline $\{\psi = k\}$ with $k \geq 0$, the horizontal velocity u is strictly deceasing for $x \in (0,\pi)$ and strictly increasing for $x \in (-\pi, 0)$.*

Proof. If $(x, y(x))$ is the parametric equation of a streamline $\psi = \psi_0$ for any $\psi_0 \in \{\psi(x,y) : (x,y) \in \overline{D_\eta}\} = \mathbb{R}^+$, then $\psi_x + \psi_y y'(x) = 0$ by definition and so $0 > y'(x) = -\frac{\psi_x}{\psi_y}$ for $x \in (0,\pi)$. By (3.5) we have

$$\frac{dy}{dx} = y' = -\frac{\psi_x}{\psi_y} = \frac{v}{u-c} = \frac{h_q}{h_p}.$$

From (3.3) we get

$$\partial_x[u(x,y(x))] = u_x + y' u_y = u_x + \frac{h_q}{h_p} u_y = \frac{1}{h_p} u_q. \tag{3.6}$$

From (2.2) and (3.5) we have

$$(c-u)u_x - v u_y = \frac{h_p u_x + h_q u_y}{h_q^2 + h_p^2} = \frac{1}{h_q^2 + h_p^2} u_q. \tag{3.7}$$

Now, the functions u and v are harmonic in the $(q,p)-$variables since the transformation (3.2) is conformal, and therefore u_q is also harmonic. Restricting ourselves to the semi-infinite rectanglar domain $\hat{D}_+ := \{(q,p) : q \in (0, c\pi), p \in (-\infty, 0)\}$, which is the image of $\{(x,y) : x \in (0,\pi), y \in (-\infty, \eta(x))\}$ under the conformal hodograph tranformation, we work as follows. By (3.5) and (3.7) we have

$$u_q = \frac{1}{(c-u)^2 + v^2}[(c-u)u_x - v u_y].$$

It follows that

$$u_q = \frac{1}{c-u} u_x = -\frac{v_y}{c-u} = 0 \text{ for } q = 0 \text{ and } q = c\pi, \tag{3.8}$$

since $v = v_y = 0$ along these sides. Along the image of the free surface $\{p = 0\}$ we have, using (3.5) and (3.6),

$$u_q = \frac{c-u}{(c-u)^2 + v^2} \partial_x[u(x,\eta(x))] < 0, \tag{3.9}$$

for $q \in [0, c\pi]$, since $\psi_y = u - c < 0$ throughout the domain and we also use (3.1). Now, if we suppose that $u_q = M > 0$ at some point (q_0, p_0) in the interior of $\hat{D}_+$, then the maximum principle for harmonic functions is contradicted. We can see this by noting the limiting boundary condition, $(u,v) \to (0,0)$ as $p \to -\infty$, and applying the maximun principle to u_q on the finite domain $\hat{D}_+^m$ with boundaries given by $\{p = 0\}, \{q = 0\}, \{q = c\pi\}, \{p = -m\}$, where m is chosen large enough that $u_q(q, -m) < M$. Therefore $u_q \leq 0$ on $\hat{D}_+$. It follows that $u_q < 0$ in the interior of $\hat{D}_+$, since otherwise given a point where u_q is zero then the maximum principle would imply that $u_q \equiv 0$ on any domain $\hat{D}_+^m$ containing that point, and so on the whole domain $\hat{D}_+$, which is a contradiction. Thus $u_q < 0$ in $\hat{D}_+$, and this inequality together with (3.6) and $h_p > 0$ ensures that u is a strictly decreasing function along any streamline in $\hat{D}_+$. The remainder of the result is proven using the fact that u is an even function in the $x-$ and the $q-$variables. □ □

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